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LETTER TO THE EDITOR

About quantum state characterization

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Abstract. Examples of different pure quantum states which are not distinguishable by the finite set of their marginal distributions are presented.

The complete information about a pure state of a quantum mechanical system is encoded in a complex-valued wavefunction $\Psi(u)$, where u usually stand for position q or momentum p. Another representation of the quantum state is possible through the Wigner distribution function $W(q, p) = \frac{1}{2\pi}\int \Psi(q - q'/2)\Psi^*(q + q'/2)\exp(-\mathrm{i}pq')\,\mathrm{d}q'$ which is a real function of position and momentum. Here atomic units with $e = m = \hbar = 1$ are used. The Wigner function takes on negative values for certain non-classical states and so cannot be interpreted as ordinary probability distribution and be measured directly. The modern method of quantum-state characterization known as phase-space tomography [1–4] is based on the measurements of the Wigner function projections, also called marginal distributions, $pr(q, \alpha) = \int W(q\cos\alpha - p\sin\alpha, q\sin\alpha + p\cos\alpha)\,\mathrm{d}p$ over the different directions α in phase space. The set of such projections in the angle interval $\alpha \in [0, \pi]$ also completely defines the quantum state. The Wigner distribution function (and therefore the wavefunction) can be reconstructed from this set by using the inverse Radon transform. A problem is that the real experimental data contain only a finite number of the Wigner function projections. In this letter we shall show that there are distinct pure states which are not distinguishable by a finite number of their marginal distributions.

Let us consider two pure quantum states defined in position representation by the wavefunctions $\Psi_1(q)$ and $\Psi_2(q)$. These wavefunctions describe the same quantum state if there is a complex number a such that $\Psi_1(q) = a\Psi_2(q)$, otherwise the quantum states are different. We are looking for different quantum states which have the same marginal distributions $pr(q, \alpha)$ for a certain number of angles $\{\alpha_i\}$. We restrict our consideration to the two states defined by the complex-conjugate wavefunctions $\Psi_1(q) = \Psi_2^*(q)$. From that, one can easily observe the equality of their position distributions: $pr_1(q, 0) = |\Psi_1(q)|^2 = |\Psi_2(q)|^2 = pr_2(q, 0)$.

For further consideration, we introduce another definition of the marginal distribution $pr(u, \alpha)$ which is related to the fractional Fourier transform (FT) $R^\alpha[\Psi(q)](u)$ of the wavefunction $\Psi(q)$ [5, 6]:

$$R^\alpha[\Psi(q)](u) = F(u, \alpha) = \int_{-\infty}^{\infty} \Psi(q) K_\alpha(q, u)\,\mathrm{d}q \tag{1}$$

with the kernel

$$K_\alpha(q,u) = \frac{1}{\sqrt{\mathrm{i}2\pi \sin\alpha}} \exp\left(\mathrm{i}\frac{(q^2+u^2)\cos\alpha - 2qu}{2\sin\alpha}\right). \tag{2}$$

The kernel of this transform is a propagator of the non-stationary Schrödinger equation for the harmonic oscillator. The fractional FT at angle $2\pi n$ (n is an integer) corresponds to the identity operator. For $\alpha = \pi/2$, relationship (1) is the ordinary FT except for a constant phase shift. One can say that the fractional FT of $\Psi(q)$ is another representation of the quantum state along an axis making some angle α with the position axis in phase space. Thus the quantum state can be described by $\Psi(q)$ in position representation that corresponds to $\alpha = 0$, or by $R^{\pi/2}[\Psi(q)](u)$ in momentum representation or by $R^{\alpha}[\Psi(q)](u)$ in arbitrary α-representation.

The marginal distribution (or Radon–Wigner transform of $\Psi(q)$) for angle α is the squared modulus of the fractional FT [7] of the wavefunction in the position representation $\Psi(q)$ where the angle is calculated from the position axis

$$pr(u,\alpha) = |R^{\alpha}[\Psi(q)](u)|^2. \tag{3}$$

For $\alpha = 0$, $\alpha = \pi/2$ and $\alpha = \pi$, the marginal distribution reduces to $|\Psi(q)|^2$, $|R^{\pi/2}[\Psi(q)](u)|^2$ and $|\Psi(-q)|^2$ respectively.

It is easy to see from (1)–(3) that the marginal distributions for complex-conjugate functions $\Psi(q)$ and $\Psi^*(q)$ enjoy the following property:

$$|R^{\alpha}[\Psi(q)](u)|^2 = |R^{-\alpha}[\Psi^*(q)](u)|^2 = |R^{\pi-\alpha}[\Psi^*(q)](-u)|^2. \tag{4}$$

So if $\Psi_1(q)$ is such that

$$|R^{\alpha}[\Psi_1(q)](u)|^2 = |R^{-\alpha}[\Psi_1(q)](u)|^2 \tag{5}$$

then the marginal distributions for angle α of $\Psi_1(q)$ and $\Psi_2(q)$ are identical: $|R^{\alpha}[\Psi_2(q)](u)|^2 \overset{(4)}{=} |R^{-\alpha}[\Psi_1(q)](u)|^2 \overset{(5)}{=} |R^{\alpha}[\Psi_1(q)](u)|^2$.

From this it follows that the quantum state need not be uniquely determined by its position $|R^0[\Psi(q)](u)|^2 = |\Psi(u)|^2$ and momentum $|R^{\pi/2}[\Psi(q)](u)|^2$ distributions [8]. Indeed, let the quantum state described in the position representation by the wavefunction $\Psi_1(q)$ have an even momentum distribution:

$$|R^{\pi/2}[\Psi_1(q)](u)|^2 = |R^{\pi/2}[\Psi_1(q)](-u)|^2.$$

Then, applying (4), we observe that the quantum state described by the wavefunction $\Psi_2(q) = \Psi_1^*(q)$ has the same position and momentum distributions: $|R^{\pi/2}[\Psi_2(q)](u)|^2 = |R^{\pi/2}[\Psi_1^*(q)](u)|^2 \overset{(4)}{=} |R^{\pi/2}[\Psi_1(q)](-u)|^2 = |R^{\pi/2}[\Psi_1(q)](u)|^2$.

It is easy to see that all even or odd complex conjugate wavefunctions $\Psi_1(q)$ and $\Psi_2(q)$ have the same position and momentum distributions. From (1), (2) follows that the fractional FT of an even or odd function satisfies

$$R^{\alpha}[\Psi(q)](u) = \pm R^{\alpha}[\Psi(-q)](u) = \pm R^{\alpha+\pi}[\Psi(q)](u) = \pm R^{\alpha}[\Psi(q)](-u) \tag{6}$$

where the $+$ sign stands for even, and the $-$ sign for odd signals. Then the marginal distributions of even and odd wavefunctions are even

$$|R^{\alpha}[\Psi(q)](u)|^2 = |R^{\alpha}[\Psi(q)](-u)|^2. \tag{7}$$

In particular $|R^{\pi/2}[\Psi(q)](u)|^2 = |R^{\pi/2}[\Psi(q)](-u)|^2$, meaning, as we have seen above, that even and odd complex-conjugate wavefunctions have the same position and momentum distributions.

It is easy to prove that relationship (5) also holds for self-fractional Fourier functions (SFFFs) $\Psi_\alpha(q)$ which are the eigenfunctions of the fractional FT operator [9–11] for some angle α:

$$R^\alpha[\Psi_\alpha(q)](u) = A\Psi_\alpha(u) \tag{8}$$

where A is a complex constant factor such that $|A| = 1$. The Hermite–Gauss mode content of such a wavefunction has been considered in [11].

As it has been shown in [12], if $\Psi_\alpha(q)$ is an α-SFFF with eigenvalue A then $\Psi_\alpha^*(q)$ is also an α-SFFF with eigenvalue A. Moreover, the fractional FT of α-SFFF with eigenvalue A for angle $-\alpha$ is given by

$$R^{-\alpha}[\Psi_\alpha(q)](u) = A^*\Psi_\alpha(u). \tag{9}$$

Then $|R^\alpha[\Psi_\alpha(q)](u)|^2 = |R^{-\alpha}[\Psi_\alpha(q)](u)|^2 = |R^\alpha[\Psi_\alpha^*(q)](u)|^2$, which corresponds to (5). It follows from the additivity property for fractional FT: $R^\alpha R^\beta = R^{\alpha+\beta}$ (see [6]) and (8) that, if a function is a SFFF for α with eigenvalue A, it is also one for αk ($k = 1, 2, \ldots$) with eigenvalue A^k and then $|R^{k\alpha}[\Psi_\alpha(q)](u)|^2 = |R^{k\alpha}[\Psi_\alpha^*(q)](u)|^2 = |\Psi_\alpha(u)|^2$. This means that the two states defined by the complex-conjugate wavefunctions $\Psi_{1,\alpha}(q) = \Psi_{2,\alpha}^*(q)$ being α-SFFF have the same marginal distributions for a sequence of angles αk, where k is an integer. All these distributions equal the position distribution. Note that in general the map of the marginal distributions for such quantum states, described by α-SFFF $\Psi_\alpha(q)$, is periodic in the angle with period α

$$|R^{k\alpha+\beta}[\Psi_\alpha(q)](u)|^2 = |R^\beta[\Psi_\alpha(q)](u)|^2. \tag{10}$$

Moreover, using the additivity property for the fractional FT, we also derive that

$$\begin{aligned} R^\beta[\Psi_\alpha(q)](u) &= R^{\alpha-(\alpha-\beta)}[\Psi_\alpha(q)](u) \\ &= AR^{\beta-\alpha}[\Psi_\alpha(q)](u) \end{aligned} \tag{11}$$

and, in particular, for $\beta = \alpha/2$ we have that the fractional FT of an α-SFFF at angles $\alpha/2$ and $-\alpha/2$ are identical except for a constant phase factor which depends on the eigenvalue:

$$R^{\alpha/2}[\Psi_\alpha(q)](u) = AR^{-\alpha/2}[\Psi_\alpha(q)](u). \tag{12}$$

Then $|R^{\alpha/2}[\Psi_\alpha(q)](u)|^2 \overset{(12)}{=} |R^{-\alpha/2}[\Psi_\alpha(q)](u)|^2 \overset{(4)}{=} |R^{\alpha/2}[\Psi_\alpha^*(q)](u)|^2$, which corresponds to (5).

So two quantum states defined by $\Psi_\alpha(q)$ and $\Psi_\alpha^*(q)$ have the same marginal distributions for the set of angles $k\alpha/2$, where k is an integer. The marginal distributions $|R^{\alpha k/2}[\Psi_\alpha(q)](u)|^2$ equal the position distribution only for even k. Note that all SFFFs for angles $2\pi/M$, where M is even, have the same momentum distribution as their complex conjugates. In particular we can treat the odd or even wavefunctions as a SFFF for angle π. Then, using (12), we come to the equality of the momentum distributions of the quantum states defined by $\Psi_\alpha(q)$ and $\Psi_\alpha^*(q)$ as has been shown above.

Finally, we can conclude that a finite number of marginal distributions, just like the position and momentum ones cannot in general completely define a quantum state.

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